The $L(\mathfrak{sl}_2)$ symmetry of the Bazhanov-Stroganov model associated with the superintegrable chiral Potts model

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Abstract

The loop algebra $L(\mathfrak{sl}_2)$ symmetry is found in a sector of the nilpotent Bazhanov-Stroganov model. The Drinfeld polynomial of a $L(\mathfrak{sl}_2)$-degenerate eigenspace of the model is equivalent to the polynomial [4, 5, 10–14] which characterizes a subspace with the Ising-like spectrum of the superintegrable chiral Potts model.

1 Introduction

The chiral Potts model is a two-dimensional solvable lattice model whose Boltzmann weights lie on a curve of genus greater than one [1–3]. Albertini, McCoy, Perk and Tang numerically found a special point on the curve where the spectra of the transfer matrix fit to an Ising-like simple form [4]. The model at the special point is called the "superintegrable" chiral Potts (SCP) model, and its free energy and interfacial tension are explicitly calculated [5].

In order to discuss Onsager's approach to the SCP model [6], let us consider the $\mathbb{Z}_N$-symmetric Hamiltonian introduced by von Gehlen and Rittenberg [7]:

$$H_{\mathrm{vGR}} = A_0 + k' A_1 = \frac{4}{N} \sum_{i=1}^{L} \sum_{m=1}^{N-1} \frac{1}{1-\omega^{-m}} \big(Z_i^{2m} + k' X_i^{-m} X_{i+1}^{m} \big). \tag{1}$$

Here, the operators $Z_i, X_i \in \mathrm{End}((\mathbb{C}^N)^{\otimes L})$ are defined by

$$\begin{aligned} Z_i(v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i} \otimes \cdots \otimes v_{\sigma_L}) &= q^{\sigma_i} v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i} \otimes \cdots \otimes v_{\sigma_L}, \\ X_i(v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i} \otimes \cdots \otimes v_{\sigma_L}) &= v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i+1} \otimes \cdots \otimes v_{\sigma_L}, \end{aligned} \tag{2}$$

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for the standard basis $\{v_\sigma | \sigma = 0, 1, \cdots, N-1\}$ of $\mathbb{C}^N$ and under the periodic boundary conditions $Z_{L+1} = Z_1$ and $X_{L+1} = X_1$. Here and hereafter we assume odd N for simplicity, and set $q = \mathrm{e}^{\frac{2\pi}{N}\sqrt{-1}}$ and $\omega = q^2$ when $q^N = 1$. The Hamiltonian H_{vGR} is derived from the expansion of the SCP transfer matrix with respect to the spectral parameter. The A_0 and A_1 in (1) satisfy the Dolan-Grady conditions $[A_i, [A_i, [A_i, A_{1-i}]]] = 16[A_i, A_{1-i}], (i = 0, 1)$ [8] and generate the Onsager algebra (OA) [6]. The Hilbert space $(\mathbb{C}^N)^{\otimes L}$ is decomposed into the direct sum of finite-dimensional irreducible representations of OA. In each of the subspaces the energy spectra of H_{vGR} fit to the Ising-like form [9]:

$$E = \alpha + k'\beta + 2\sum_{i=1}^{n} m_i \sqrt{1 + 2k'\cos\theta_i + k'^2}, \quad m_i \in \{-l_i, -l_i+2, \cdots, l_i\}. \tag{3}$$

Here, $\alpha, \beta \in \mathbb{R}$ and $\theta_i \in \mathbb{R}/2\pi\mathbb{R}$ are such parameters that are determined not only by the OA approach.

The parameters α, β and θ_i in the spectra (3) are determined by the functional relations of the SCP transfer matrix. Introduce [4, 5, 10–14]

$$P_{\mathrm{CP}}(\xi^N) = \omega^{-p_b} \sum_{j=0}^{N-1} \frac{(1-\xi^N)^L (\xi\omega^j)^{-p_a-p_b}}{(1-\xi\omega^j)^L F_{\mathrm{CP}}(\xi\omega^j) F_{\mathrm{CP}}(\xi\omega^{j+1})}, \tag{4}$$

and call it the chiral Potts polynomial. Here $F_{\mathrm{CP}}(\xi) := \prod_{i=1}^{R}(1 + \xi u_i \omega)$ and $\{u_i\}$ satisfy the following coupled nonlinear equations:

$$\left(\frac{u_i + \omega^{-1}}{u_i + \omega^{-2}}\right)^L = \omega^{-p_a-p_b-1} \prod_{\substack{j=1 \\ j(\neq i)}}^{R} \frac{u_i - u_j\omega^{-1}}{u_i - u_j\omega}, \tag{5}$$

and p_a and p_b are chosen so that $P_{\mathrm{CP}}(0)$ is finite and non-zero. The coupled nonlinear equations (5) give the pole-free conditions for $P_{\mathrm{CP}}(\zeta)$. If the polynomial $P_{\mathrm{CP}}(\zeta)$ is factorized as $P_{\mathrm{CP}}(\zeta) = \prod_{i=1}^{n}(1 - \zeta_i^{-1}\zeta)^{l_i}$ with distinct zeros $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$, the parameters $\{\theta_i\}$ in (3) are determined through $\zeta_i = -\tan(\theta_i/2)$ and the dimensions of the corresponding representation space of OA are given by $\prod_{i=1}^{n}(l_i + 1)$ [9]. The chiral Potts polynomial $P_{\mathrm{CP}}(\zeta)$ is first obtained in the special cases: the sector $R = 0$ [5] and the case $N = 3$ [4]. The expression (4) for $P_{\mathrm{CP}}(\zeta)$ with general N and R is given by Baxter [14]. We remark that the results in [4, 5, 10, 14] imply $l_i = 1$ for all i, that is, the dimensionality of the OA representation space is $2^{\deg P_{\mathrm{CP}}(\zeta)}$ where $\deg P_{\mathrm{CP}}(\zeta)$ denotes the degree of $P_{\mathrm{CP}}(\zeta)$.

The chiral Potts polynomial $P_{\mathrm{CP}}(\zeta)$ plays the central role in the spectra (3). However, its definition is still nontrivial at least algebraically. In fact, it is based on the functional relations for the transfer matrices of chiral Potts model [15], which are rather complicated, and any mathematical background has not been explicitly discussed, yet. We thus want to determine the parameters $\{\theta_i\}$ only by OA as is established in the case of 2D Ising model [6]. However, the representation theory of OA has not been fully developed. For instance, all the finite-dimensional representations are not classified, yet. On the other hand, it is known that the OA is isomorphic to a subalgebra of the $\mathfrak{sl}_2$-loop algebra, $L(\mathfrak{sl}_2)$ [16]. It is thus natural to ask whether the polynomial $P_{\mathrm{CP}}(\zeta)$ can be understood in terms of the representation theory of $L(\mathfrak{sl}_2)$ [17, 18]. Our main purpose here is to discuss the polynomial $P_{\mathrm{CP}}(\zeta)$ from an algebraic point of view. The result might be useful to develop the representation theory of OA in order to determine the parameters $\{\theta_i\}$.

Bazhanov and Stroganov introduced an integrable N-state spin chain, which connects the chiral Potts model to the six-vertex model [19]. At the superintegrable point, the model is called nilpotent Bazhanov-Stroganov (NBS) model and is equivalent to an XXZ-type spin chain at $q^N = 1$. It has therefore Bethe eigenstates. The NBS transfer matrix $\tau_{\rm NBS}(z)$ commutes with the transfer matrix of the SCP model as well as with the von Gehlen-Rittenberg's Hamiltonian $H_{\rm vGR}$ [19]. However, not all the Bethe states of $\tau_{\rm NBS}(z)$ are eigenstates of $H_{\rm vGR}$: as shown in [5, 10], from a given Bethe eigenstate of $\tau_{\rm NBS}(z)$, the Hamiltonian $H_{\rm vGR}$ generates such a subspace that has the Ising-like spectra (3), while it is nontrivial to construct a complete set of eigenvectors of $H_{\rm vGR}$ in the subspace. Furthermore, the subspace gives a degenerate eigenspace of $\tau_{\rm NBS}(z)$ with respect to the OA symmetry. Here we note that the commutativity $[H_{\rm vGR}, \tau_{\rm NBS}(z)] = 0$ leads to the OA symmetry of the NBS model.

We now discuss the Ising-like spectra of the SCP model from the viewpoint of the $L(\mathfrak{sl}_2)$ symmetry. Recently it is found that the XXZ spin chain at $q^N = 1$ has large degeneracies in the energy spectra and the degeneracies are described by the $L(\mathfrak{sl}_2)$ symmetry [20]. In this letter, we show that the NBS model has the $L(\mathfrak{sl}_2)$ symmetry in a certain sector. It indeed gives a "higher" symmetry than the OA symmetry. Applying the approach of [21] to a Bethe state of $\tau_{\rm NBS}(z)$ in the sector, we obtain the Drinfeld polynomial if the zeros are distinct [22]. We then find that the Drinfeld polynomial is equivalent to the chiral Potts polynomial $P_{\rm CP}(\zeta)$ of the Ising-like spectra associated with the Bethe state. Therefore, the representation space of OA for the polynomial $P_{\rm CP}(\zeta)$ has the same dimensions as the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the NBS model $\tau_{\rm NBS}(z)$.

The letter consists of the following: in §2 we introduce the NBS model; in §3 we derive the $L(\mathfrak{sl}_2)$ symmetry of the NBS model in a sector; in §4 we show that the Drinfeld polynomial is equivalent to the chiral Potts polynomial $P_{\rm CP}(\zeta)$ in the sector, and finally we give a conjecture.

2 Nilpotent Bazhanov-Stroganov model

First we introduce an XXZ-type spin chain defined for generic q. Later the model is identified with the NBS model when $q^N = 1$. The L-operators $\mathcal{L}_i(z) \in \mathrm{End}(\mathbb{C}^2 \otimes (\mathbb{C}^N)^{\otimes L})$, $(i = 1, \dots, L)$ for the model are given by

$$\mathcal{L}_i(z) := \begin{pmatrix} q^{-\frac{1}{2}}(z\hat{k}_i^{\frac{1}{2}} - z^{-1}\hat{k}_i^{-\frac{1}{2}}) & (q - q^{-1})\hat{f}_i \\ (q - q^{-1})\hat{e}_i & q^{\frac{1}{2}}(z\hat{k}_i^{-\frac{1}{2}} - z^{-1}\hat{k}_i^{\frac{1}{2}}) \end{pmatrix}. \tag{6}$$

Here $\{\hat{k}_i, \hat{e}_i, \hat{f}_i\}$ is the N-dimensional irreducible representation of the quantum group $U_q(\mathfrak{sl}_2)$ acting on the ith component of the tensor product $(\mathbb{C}^N)^{\otimes L}$. We construct the monodromy matrix $\mathcal{T}(z)$ and the transfer matrix $\tau(z)$ as

$$\mathcal{T}(z) := \prod_{i=1}^{\overset{\curvearrowleft}{L}} \mathcal{L}_i(z) =: \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}, \quad \tau(z) := \mathrm{tr}_{\mathbb{C}^2}(\mathcal{T}(z)) = A(z) + D(z),$$

where $A(z), B(z), C(z), D(z) \in \mathrm{End}((\mathbb{C}^N)^{\otimes L})$. It is well-known that the transfer matrix $\tau(z)$ forms a commutative family, $\tau(z)\tau(w) = \tau(w)\tau(z)$. Since the $\mathcal{T}(z)$ is intertwined by the R-matrix of the six-vertex model, the operators $A(z), B(z), C(z)$ and $D(z)$ satisfy the same relations as those in the case of the spin-1/2 XXZ spin chain [23]. Then it is straightforward

to apply the algebraic Bethe ansatz to the XXZ-type spin chain (6). Let $|0\rangle := v_0 \otimes \cdots \otimes v_0 \in (\mathbb{C}^N)^{\otimes L}$ be the reference state. One sees

$$\begin{aligned} &A(z)|0\rangle = q^{-\frac{L}{2}}(zq^{\frac{N-1}{2}} - z^{-1}q^{-\frac{N-1}{2}})^L|0\rangle, \quad D(z)|0\rangle = q^{\frac{L}{2}}(zq^{-\frac{N-1}{2}} - z^{-1}q^{\frac{N-1}{2}})^L|0\rangle, \\ &C(z)|0\rangle = 0. \end{aligned}$$

If a set of variables $\{z_i | i = 1, \dots, R\}$ satisfies the Bethe equations

$$\left(\frac{z_i^2 q^{N-1} - 1}{z_i^2 - q^{N-1}}\right)^L = q^L \prod_{\substack{j=1 \\ j(\neq i)}}^{R} \frac{z_i^2 q^2 - z_j^2}{z_i^2 - z_j^2 q^2}, \tag{7}$$

the Bethe state $|R; \{z_i\}\rangle := \prod_{i=1}^R B(z_i)|0\rangle$ gives an eigenstate of the transfer matrix $\tau(z)$.

Secondly we see that, at $q^N = 1$, the XXZ-type spin chain (6) reduces to the NBS model, i.e., $\tau(z)|_{q^N=1} = \tau_{\mathrm{NBS}}(z)$. The L-operators (6) are rewritten as

$$\mathcal{L}_i(z) = \begin{pmatrix} -zq^{-1}Z_i^{-1} + z^{-1}Z_i & -(Z_i^{-1} - Z_i)X_i \\ X_i^{-1}(Z_i^{-1} - Z_i) & z^{-1}Z_i^{-1} - zqZ_i \end{pmatrix}, \tag{8}$$

through the following nilpotent representation of $U_q(\mathfrak{sl}_2)$ at $q^N = 1$:

$$\hat{k}_i^{\frac{1}{2}} = -q^{-\frac{1}{2}}Z_i^{-1}, \quad \hat{e}_i = X_i^{-1}\frac{Z_i^{-1} - Z_i}{q - q^{-1}}, \quad \hat{f}_i = \frac{Z_i - Z_i^{-1}}{q - q^{-1}}X_i. \tag{9}$$

Recall that the Z_i and X_i are the operators defined at $q^N = 1$ in (2). The L-operators (8) are equivalent to the original ones [19] at superintegrable point: we take the principal gradation [24] and make an appropriate similarity transformation, then we obtain the expression (8). Since the representation (9) is called nilpotent in contrast to the cyclic representation of $U_q(\mathfrak{sl}_2)$ at $q^N = 1$, we have referred to the model as the nilpotent BS model. One also notices that, through the change of variables $z_i \mapsto -q^3 u_i$, the Bethe equations (7) are identified with the coupled nonlinear equations (5) with $q^N = 1$ and some specific p_a and p_b. In fact, at $q^N = 1$, the Bethe state $|R; \{z_i\}\rangle$ is shown to belong to the subspace with the spectra (3) characterized by $P_{\mathrm{CP}}(\zeta)$ (4) [10].

3 Loop algebra $L(\mathfrak{sl}_2)$ symmetry

We show that the NBS model has a loop algebra $L(\mathfrak{sl}_2)$ symmetry. We first consider the operators $A(z), B(z), C(z)$ and $D(z)$ with generic q and then take the limit $q^N \to 1$ to discuss the NBS model. Introduce

$$\begin{aligned} A &:= \lim_{z\to\infty} \frac{A(z)}{z^L q^{-\frac{L}{2}}} = \lim_{z\to\infty} \frac{D(z)}{z^{-L} q^{\frac{L}{2}}} = \underbrace{\hat{k}^{\frac{1}{2}} \otimes \cdots \otimes \hat{k}^{\frac{1}{2}}}_{L}, \\ B_\pm &:= \lim_{z^{\pm 1}\to\infty} \frac{B(z)}{n_\pm(z)} = q^{-\frac{1}{2}(L+1)} \sum_{i=1}^{L} q^i \underbrace{\hat{k}^{\mp\frac{1}{2}} \otimes \cdots \otimes \hat{k}^{\mp\frac{1}{2}}}_{i-1} \otimes \hat{f} \otimes \underbrace{\hat{k}^{\pm\frac{1}{2}} \otimes \cdots \otimes \hat{k}^{\pm\frac{1}{2}}}_{L-i}, \\ C_\pm &:= \lim_{z^{\pm 1}\to\infty} \frac{C(z)}{n_\pm(z)} = q^{\frac{1}{2}(L+1)} \sum_{i=1}^{L} q^{-i} \underbrace{\hat{k}^{\pm\frac{1}{2}} \otimes \cdots \otimes \hat{k}^{\pm\frac{1}{2}}}_{i-1} \otimes \hat{e} \otimes \underbrace{\hat{k}^{\mp\frac{1}{2}} \otimes \cdots \otimes \hat{k}^{\mp\frac{1}{2}}}_{L-i}, \end{aligned}$$

with normalization factors $n_\pm(z) = (\pm z^{\pm 1})^{L-1}(q - q^{-1})$. Through the relations among $A(z), B(z), C(z)$ and $D(z)$, we find that the operators

$$k_0^{-1} = k_1 = A^2, \quad e_0 = B_+, \quad e_1 = C_+, \quad f_0 = C_-, \quad f_1 = B_-,$$

give a finite-dimensional representation of the quantum affine algebra $U_q'(\hat{\mathfrak{sl}_2})$. After a simple calculation, we obtain

$$(B_\pm)^m = \sum_{\substack{0\le\lambda_i\le m\\ \lambda_1+\cdots+\lambda_L=m}} q^{\sum_j\left(j-\frac{1}{2}(L+1)\right)\lambda_j} \frac{[m]!}{[\lambda_1]!\cdots[\lambda_L]!} \bigotimes_{i=1}^{L} \hat{f}^{\lambda_i}\hat{k}^{\pm\frac{1}{2}\left(\sum_{j(<i)}-\sum_{j(>i)}\right)\lambda_j},$$

$$(C_\pm)^m = \sum_{\substack{0\le\lambda_i\le m\\ \lambda_1+\cdots+\lambda_L=m}} q^{-\sum_j\left(j-\frac{1}{2}(L+1)\right)\lambda_j} \frac{[m]!}{[\lambda_1]!\cdots[\lambda_L]!} \bigotimes_{i=1}^{L} \hat{k}^{\mp\frac{1}{2}\left(\sum_{j(<i)}-\sum_{j(>i)}\right)\lambda_j}\hat{e}^{\lambda_i},$$

where $[m] := \frac{q^m - q^{-m}}{q - q^{-1}}$ and $[m]! := \prod_{i=1}^m [i]$.

We now consider the limit $q^N \to 1$. One easily finds from the expression above that, if we set $q^N = 1$, then $(B_\pm)^N = (C_\pm)^N = 0$. Define

$$\begin{aligned} H^{(n)} &:= \frac{1}{n}\sum_{i=1}^{L} \underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{i-1}\otimes\hat{h}\otimes\underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{L-i}, \\ B_\pm^{(n)} &:= \lim_{q^N\to 1}\frac{(B_\pm)^n}{[n]!}, \qquad C_\pm^{(n)} := \lim_{q^N\to 1}\frac{(C_\pm)^n}{[n]!}, \end{aligned} \tag{10}$$

where $\hat{h} = \mathrm{diag}\{N-1, N-3, \ldots, -N+1\} \in \mathrm{End}(\mathbb{C}^N)$. By considering the relations among $A(z), B(z), C(z)$ and $D(z)$ in the limit $q^N \to 1$, we have

$$\begin{aligned} [\tau_{\mathrm{NBS}}(z), B_\pm^{(N)}] &= -z^{\pm 1} B_\pm^{(N-1)} B(z)(q^{-\frac{L}{2}}A^{\pm 1} - q^{\frac{L}{2}}A^{\mp 1}), \\ [\tau_{\mathrm{NBS}}(z), C_\pm^{(N)}] &= z^{\pm 1} C_\pm^{(N-1)} C(z)(q^{-\frac{L}{2}}A^{\pm 1} - q^{\frac{L}{2}}A^{\mp 1}). \end{aligned}$$

Recall that $\tau(z)|_{q^N=1} = \tau_{\mathrm{NBS}}(z)$. Then we have $[\tau_{\mathrm{NBS}}(z), B_+^{(N)}] = [\tau_{\mathrm{NBS}}(z), C_+^{(N)}] = 0$ in the sector $A^2 = q^L$ and $[\tau_{\mathrm{NBS}}(z), B_-^{(N)}] = [\tau_{\mathrm{NBS}}(z), C_-^{(N)}] = 0$ in the sector $A^2 = q^{-L}$.

From the relations among $\{A, B_\pm, C_\pm\}$ and the general theory of quantum affine algebras at roots of unity [25], we show

$$\begin{aligned} &[B_+^{(N)}, B_-^{(N)}] = [C_+^{(N)}, C_-^{(N)}] = 0, \\ &[H^{(N)}, B_\pm^{(N)}] = -2B_\pm^{(N)}, \qquad [H^{(N)}, C_\pm^{(N)}] = 2C_\pm^{(N)}, \\ &[B_\pm^{(N)}, [B_\pm^{(N)}, [B_\pm^{(N)}, C_\pm^{(N)}]]] = 0, \qquad [C_\pm^{(N)}, [C_\pm^{(N)}, [C_\pm^{(N)}, B_\pm^{(N)}]]] = 0. \end{aligned}$$

Furthermore, in the sector with $A^2 = 1$ in $(\mathbb{C}^N)^{\otimes L}$, we have

$$[B_\pm^{(N)}, C_\mp^{(N)}] = \pm H^{(N)}.$$

Let $\{H_i, E_i, F_i | i = 0, 1\}$ express the Chevalley generators of the $\mathfrak{sl}_2$-loop algebra $L(\mathfrak{sl}_2)$. The map $\hat{}: L(\mathfrak{sl}_2) \to \mathrm{End}((\mathbb{C}^N)^{\otimes L})$ defined by

$$\hat{H}_0 = -\hat{H}_1 := -H^{(N)}, \quad \hat{E}_0 := B_+^{(N)}, \quad \hat{E}_1 := C_+^{(N)}, \quad \hat{F}_0 := C_-^{(N)}, \quad \hat{F}_1 := B_-^{(N)},$$

gives a finite-dimensional representation of the loop algebra $L(\mathfrak{sl}_2)$ in the sector with $A^2 = 1$. Thus we have the following:

Proposition 3.1 (cf. [20]). *If $q^N = 1$ and L is a multiple of N, the transfer matrix $\tau_{\mathrm{NBS}}(z)$ has the $L(\mathfrak{sl}_2)$ symmetry in the sector with $A^2 = 1$:*

$$[\tau_{\mathrm{NBS}}(z), \hat{H}_{0,1}] = [\tau_{\mathrm{NBS}}(z), \hat{E}_{0,1}] = [\tau_{\mathrm{NBS}}(z), \hat{F}_{0,1}] = 0.$$

4 $L(\mathfrak{sl}_2)$-degenerate eigenspaces and the SCP spectra

We now discuss the dimensions of $L(\mathfrak{sl}_2)$-degenerate eigenspaces of the NBS model. Let us express the highest weight conditions of the Drinfeld realization of $L(\mathfrak{sl}_2)$ in terms of the Chevalley generators [17, 18]. We call a vector Ω a highest weight vector if it is annihilated by E_1 and F_0, $E_1\Omega = F_0\Omega = 0$, and is diagonalized by $H_1(= -H_0)$, $(E_1)^k(E_0)^k$ and $(F_0)^k(F_1)^k$ for $k \in \mathbb{Z}_{\geq 0}$. If a representation is generated by a highest weight vector, we call it highest weight. For a finite-dimensional representation generated by highest weight vector Ω, we define the following polynomial:

$$P_{\mathrm{D}}(\xi) = \sum_{k=0}^{n} \lambda_k (-\xi)^k,$$

where n and λ_k denote the eigenvalues of H_1 and $(E_1)^k(E_0)^k/(k!)^2$ on Ω, respectively. (See eq. (4.2) of [26].) If the zeros of $P_{\mathrm{D}}(\xi)$ are distinct, the representation is irreducible [22]. And the polynomial $P_{\mathrm{D}}(\xi)$ is called the Drinfeld polynomial.

Consider the Bethe states of the NBS model in the sector with $A^2 = q^{-L-2R} = 1$. Here we recall that both L and R are multiples of N. Let $\{z_i\}$ be a set of regular solutions of the Bethe equations (7) at $q^N = 1$. Here, if solutions of (7), $\{z_i\}$, are finite, distinct and nonzero, we call them regular. In the similar way to [21], it is shown that the Bethe state $|R; \{z_i\}\rangle$ is a highest weight vector. After some calculation, we obtain

$$\begin{aligned}
&\hat{H}_1 |R; \{z_i\}\rangle = \frac{L(N-1) - 2R}{N} |R; \{z_i\}\rangle, \\
&\frac{(\hat{E}_1)^k}{k!} \frac{(\hat{E}_0)^k}{k!} |R; \{z_i\}\rangle = (-)^k \chi_{kN} |R; \{z_i\}\rangle, \qquad \left(0 \leq k \leq \frac{L(N-1) - 2R}{N}\right),
\end{aligned}$$

where χ_k is defined by the following series expansion:

$$\frac{\prod_{i=1}^{N-1} \phi(xq^{2i-N})}{F(xq)F(xq^{-1})} = \frac{(1-x^N)^L}{(1-x)^L F(xq) F(xq^{-1})} = \sum_{k=0}^{\infty} \chi_k x^k. \tag{11}$$

The functions $\phi(x)$ and $F(\xi)$ are given by $\phi(x) := (1-x)^L$ and $F(\xi) := \prod_{i=1}^{R}(1 - \xi z_i)$, respectively. The result is summarized as follows:

Proposition 4.1. *If L and R are multiples of N, the Bethe state of the NBS model, $|R; \{z_i\}\rangle$, is highest weight. The polynomial $P_{\mathrm{D}}(\zeta)$ of the finite-dimensional representation generated by $|R; \{z_i\}\rangle$ is given by*

$$P_{\mathrm{D}}(\xi^N) = \sum_{k=0}^{\frac{L(N-1)-2R}{N}} \chi_{kN} \xi^{kN} = N \sum_{j=0}^{N-1} \frac{(1-\xi^N)^L}{(1-\xi q^{2j})^L F(\xi q^{2j-1}) F(\xi q^{2j+1})}. \tag{12}$$

The polynomial $P_{\rm D}(\zeta)$ gives the Drinfeld polynomial if the zeros are distinct. In general, if the Drinfeld polynomial of a finite-dimensional irreducible representation is factorized as $P_{\rm D}(\zeta) = \prod_{i=1}^{n}(1-\zeta_i^{-1}\zeta)^{l_i}$ with distinct zeros $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$, the dimensions of the representation are given by $\prod_{i=1}^{n}(l_i+1)$. Thus, in the case $l_i = 1$ for all i, the dimensionality of the $L(\mathfrak{sl}_2)$-degenerate eigenspace of $\tau_{\rm NBS}(z)$ is $2^{\deg P_{\rm D}(\zeta)}$.

Comparing expression (4) with (12) we have the following:

Proposition 4.2. *If the zeros of the chiral Potts polynomial $P_{\rm CP}(\zeta)$ (4) are distinct, the $P_{\rm CP}(\zeta)$ with $p_a + p_b = 0$ is equivalent to the Drinfeld polynomial $P_{\rm D}(\zeta)$ in* (12).

It is suggested from proposition 4.2 that the $2^{\deg P_{\rm CP}(\zeta)}$-dimensional representation space of OA characterized by the polynomial $P_{\rm CP}(\zeta)$ corresponds to the degenerate eigenspace of $L(\mathfrak{sl}_2)$ of the Drinfeld polynomial $P_{\rm D}(\zeta)$. Here we remark that we have verified this conjecture in the case of $L = N = 3$.

The representation space of OA of the chiral Potts polynomial $P_{\rm CP}(\zeta)$ and the $L(\mathfrak{sl}_2)$-degenerate eigenspace of the Drinfeld polynomial $P_{\rm D}(\zeta)$ have the same dimensions. Furthermore, they have the same Bethe state. As is shown in [5, 10], the subspace characterized by $P_{\rm CP}(\zeta)$ is generated by repeated application of the SCP transfer matrix to the Bethe state. The conjecture is derived if we assume that Bethe states of $\tau_{\rm NBS}(z)$ are complete and also that the Bethe states are nondegenerate with respect to eigenvalues of $\tau_{\rm NBS}(z)$.

We have clarified one of the fundamental algebraic aspects of the polynomial $P_{\rm CP}(\zeta)$ characterizing the Ising-like spectrum of SCP model. One notices that the $L(\mathfrak{sl}_2)$ symmetry of the NBS model readily provides an OA symmetry of the model. We thus speculate that, in a certain sector, the OA symmetry should be the origin of the OA structure of both SCP model and von Gehlen-Rittenberg's model $H_{\rm vGR}$. However, it is our future problem to discuss the representation of A_0 and A_1 in terms of the $L(\mathfrak{sl}_2)$ representation.

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